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# A CHARACTERIZATION OF REDUCED INCIDENCE ALGEBRAS

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This paper aims to give a criterium, in terms of the partial ordering on the po-set only, which decides whether or not an equivalence relation on the segments of the po-set is compatible (defined by Doubilet, Rota & Stanley [2] in terms of the convolution and the functions of the incidence algebra).

## Introduction

Let $P$ be a locally finite partially ordered set. Let $K$ be a field. Then the incidence algebra $I(P)$ of $P$ is defined as the algebra consisting of all functions $f: P^2 \to K$ with $f(x, y) \neq 0$ only if $x \leq y$ (see Rota [1] and Doubilet, Rota and Stanley [2]). An equivalence relation on $p^2$ is called compatible iff the functions $f \in I(P)$ which are constant on equivalence classes form a subalgebra (see [2]).

Now the question arises if the compatible equivalence relations might be characterized only in terms of $P$ and the ordering. Smith gave a sufficient condition in [3] but Doubilet, Rota and Stanley indicated in [2] that in general no simple criterion exists.

The aim of this paper is to give a characterization which is indeed quite similar to Smith's sufficient but not necessary condition.

## Definitions

Let $P$ be a fixed locally finite poset (i.e.: a set with a relation $\leq$ which is reflexive, transitive and antisymmetric such that all segments $[x, y] = \{z \in P : x \leq z \leq y\}$ are finite). Let $K$ be a fixed field.

Let $\operatorname{seg}(P) = \{(x, y) \in P^2 : x \leq y\}$. Then $I(P) = \{f : \operatorname{seg}(P) \to K\}$ together with pointwise vectorspace structure and convolution

$$(f * g)(x, y) = \sum_{z \in [x,y]} f(x, z)g(z, y)$$

is called the incidence algebra of $P$.

Let $\sim$ be an equivalence relation on $\operatorname{seg}(P)$. The reduced incidence "algebra" is then defined by

$$R(P, \sim) = \{f \in I(P) : (x, y) \sim (u, v) \Rightarrow f(x, y) = f(u, v)\}.$$

It is obvious that $R(P, \sim)$ is a subvectorspace of $I(P)$. In general however, it is not necessarily a subalgebra. This gives rise to the following definition:

An equivalence relation $\sim$ on seg$(P)$ is called compatible iff $R(P, \sim)$ is closed under convolution.

## Result

The aim of this paper is to prove the following

**Theorem.** *Let $\sim$ be an equivalence relation on* seg$(P)$. *Then $\sim$ is compatible iff for all $(x, y) \sim (u, v)$ there exists a bijection $\varphi : [x, y] \to [u, v]$ such that for all $z \in [x, y]$: $(x, z) \sim (u, \varphi(z))$ and $(z, y) \sim (\varphi(z), v)$.*

**Proof.** ($\Leftarrow$). We have to show that $f, g \in R(P, \sim)$ implies $f * g \in R(P, \sim)$.

Let $(x, y) \sim (u, v)$. Then there exists a bijection $\varphi$ with the property above which gives:

$$\begin{aligned} f * g(u, v) &= \sum_{w \in [u,v]} f(u, w) g(w, v) \quad \text{(by definition)} \\ &= \sum_{z \in [x,y]} f(u, \varphi(z)) g(\varphi(z), v) \quad \text{(since } \varphi \text{ is a bijection)} \\ &= \sum_{z \in [x,y]} f(x, z) g(z, y) \quad \text{(property of } \varphi\text{; } f, g \text{ are constant on equivalent segments)} \\ &= f * g(x, y) \quad \text{(by definition).} \end{aligned}$$

Therefore $f * g$ is constant on equivalence classes, i.e. $f * g \in R(P, \sim)$.

($\Rightarrow$). Let $(x, y) \sim (u, v)$. We have to show that there is a bijection $\varphi$ with the required property.

First we will show that $[x, y]$ and $[u, v]$ have the same cardinality:

Let $\zeta(x, y) = 1$ for all $x, y$ $(x \leq y)$. Obviously $\zeta \in R(P, \sim)$ so $\zeta * \zeta \in R(P, \sim)$ and we have:

$$|[x, y]| = \sum_{z \in [x,y]} 1 = \zeta * \zeta(x, y) = \zeta * \zeta(u, v) = \sum_{w \in [u,v]} 1 = |[u, v]|.$$

Now for $z_0 \in [x, y]$ let $f$, $g$ be the characteristic functions on the equivalence class of $(x, z_0)$, $(z_0, y)$. I.e.

$$f(a, b) = \begin{cases} 1 & \text{iff } (a, b) \sim (x, z_0) \\ 0 & \text{else} \end{cases}.$$

Again we have $f, g \in R(P, \sim)$ and therefore $f*g(x, y) = f*g(u, v)$. But

$$f*g(x, y) = \sum_{z\in[x,v]} f(x, z)g(z, y) = E_1.$$

Define $z \approx z_0$ iff $(x, z) \sim (x, z_0)$ and $(z, y) \sim (z_0, y)$. So

$$E_1 = \sum_{\substack{z\in[x,y]\\ z\approx z_0}} 1 = |\{z \in [x, y] : z \approx z_0\}|$$

(Because $fg$ is concentrated only on these values of $z$). And

$$f*g(u, v) = \sum_{w\in[u,v]} f(u, w)g(w, v) = E_2.$$

Define $z_0 \mathbb{R}\, w$ iff $(u, w) \sim (x, z_0)$ and $(w, v) \sim (z_0, y)$. So

$$E_2 = |\{w \in [u, v] : z_0 \mathbb{R}\, w\}|.$$

Now we might choose a bijection

$$\varphi_{z_0} : \{z \in [x, y] : z \approx z_0\} \to \{w \in [u, v] : z_0 \mathbb{R}\, w\}.$$

Let $\varphi$ be pieced together from the $\varphi_{z_0}$'s. This means we take fixed representatives $z_0$ out of each equivalence class of $\approx$ (which is indeed an equivalence relation on $[x, y]$) and define $\varphi(z) = \varphi_{z_0}(z)$ iff $z \approx z_0$.

Then $\varphi$ has the required property because each $\varphi_{z_0}$ has it, i.e. $z \approx z_0 \Rightarrow z \mathbb{R}\, \varphi_{z_0}(z) = \varphi(z)$.

Furthermore $\varphi$ is one to one: Let $\varphi(z) = \varphi(z')$. Then $z \mathbb{R}\, \varphi(z)$ and $z' \mathbb{R}\, \varphi(z') = \varphi(z)$ and therefore $z \approx z'$. But restricted to an equivalence class, $\varphi$ is a $\varphi_{z_0}$, and this therefore one to one. Thus $z = z'$.

We have proved that $[x, y]$ and $[u, v]$ have the same finite cardinality and therefore $\varphi$ is a bijection. This ends the proof.

**Remark**

If $(x, y) \sim (u, v)$ then there exists a bijection $\varphi$ by the theorem such that for all $z \in [x, y]$, $z \mathbb{R}\, \varphi(z)$. Specifically $(x, z) \sim (u, \varphi(z))$ and therefore again by the theorem there is a bijection $\varphi_z : [x, z] \to [u, \varphi(z)]$. We might ask if it is possible to choose $\varphi$ in such a way that for all $z \in [x, y]$, $\varphi$ restricted to $[x, z]$ fulfills the condition of the theorem for $(x, z) \sim (u, \varphi(z))$.

Suppose this is possible. Let $x \leqslant z' \leqslant z \leqslant y$. Then $\varphi(z) \geqslant \varphi_z(z') = \varphi(z')$, so $\varphi$ is monotonic! Let $u \leqslant \varphi(z') \leqslant \varphi(z) \leqslant v$. Then there exists a $z'' \leqslant z$ with $\varphi(z') \geqslant \varphi_z(z'') = \varphi(z'')$ ($\varphi_z$ is onto). Because $\varphi$ is injective, $z \geqslant z'' = z'$ and so $\varphi$ is a po-isomorphism (i.e. isotonic bijection). This would show that $\sim$ is a finer relation than the compatible equivalence relation "po-isomorphic". But Doubilet, Rota & Stanley gave an example in [2] that not all compatible equivalence relations are finer than the relation "po-isomorphic".

So we see that such a special choice of $\varphi$ is in general not possible!

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